

Amplitude equations near pattern forming instabilities for strongly driven ferromagnets

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Abstract

A transversally driven isotropic ferromagnet being under the influence of a static external and an uniaxial internal anisotropy field is studied. We consider the dissipative Landau-Lifshitz equation as the fundamental equation of motion and treat it in $1 + 1$ dimensions. The stability of the spatially homogeneous magnetizations against inhomogeneous perturbations is analyzed. Subsequently the dynamics above threshold is described via amplitude equations and the dependence of their coefficients on the physical parameters of the system is determined explicitly. We find soft- and hard-mode instabilities, transitions between sub- and supercritical behaviour, various bifurcations of higher codimension, and present a series of explicit bifurcation diagrams. The analysis of the codimension-2 point where the soft- and hard-mode instabilities coincide leads to a system of two coupled Ginzburg-Landau equations.

1 Introduction

By now there is a huge amount of work treating pattern formation in magnetically ordered substances under equilibrium conditions, i.e. under the influence of static external as well as internal fields. It has led to what seems to be both theoretically and experimentally a rather complete picture. The situation is much more precarious if the system is driven far from equilibrium by applying strong oscillating magnetic fields. Being hailed as a canonical example for pattern formation by some authors (comp. e.g. [1]), critical questions have been raised by others [2].

The treatment of ferromagnetic resonance phenomena, which has been initiated by Suhl [3] and uses spin waves and their nonlinear interactions, has proved to be an important development in this field. Truncating the number of relevant modes properly many results have been obtained which explain most of the observed phenomena pretty successfully. They comprise the occurrence of threshold values for parametric instabilities as well as typical nonlinear effects including various bifurcation scenarios to low dimensional chaos.

Even the use of the full (nontruncated) set of spin wave equations is doubtful, however, if one discusses pattern formation for large aspect ratios in spatially extended systems, because it lacks a systematic derivation. Consequently Elmer [4] has proposed another approach. Starting from the fundamental Landau-Lifshitz equation of motion describing the dynamics of the magnetization field on a mesoscopic level and including various damping mechanisms (see below) he derived, for fixed values of the system parameters, amplitude equations by using multiple-scale perturbation theory. His main objective was to discuss his findings as functions of the strength of the driving field.

We choose as our starting point the dissipative Landau-Lifshitz equation

$$\dot{\vec{S}}(x, t) = - \left(\vec{S} \times \vec{H}_{eff} + \Gamma \vec{S} \times \left(\vec{S} \times \vec{H}_{eff} \right) \right). \quad (1)$$

containing a Gilbert damping term on the right hand side as well. The latter has been justified microscopically in [5, 6]. Disregarding the dipolar interaction as far as it cannot be incorporated into a local anisotropy field we study it adopting a rather different point of view. Based on a work by Träxler et al. [7], where it has been shown that the homogeneous solutions of equation (1) possess a surprisingly rich bifurcation scenario when the physical parameters of the system are varied, we extend their work by admitting spatial inhomogeneities. As in their article we keep the amplitude of the driving field fixed and use the static external field together with the detuning from resonance as fundamental parameters. Aiming at obtaining a complete overview in the space of parameters we find sub- and supercritical soft-mode and, for the first time for this magnetic system, hard-mode instabilities also. The corresponding amplitude equations are derived together with all of their coefficients explicitly.

We then proceed to show that our results imply the existence of some what we deem to be highly interesting bifurcations of higher codimension. We just mention here the codimension-2 example discussed in section 5.3 which arises when the soft- and hard-mode instabilities coincide and which leads to a system of two coupled real and complex Ginzburg-Landau equations.

However, in our opinion the results given here represent a contribution to earning strongly driven ferromagnets a solid place among the classic pattern forming forebears in hydrodynamics, nonlinear optics, etc. [2].

2 Equation of motion

We start from the dissipative Landau-Lifshitz equation of motion (1) and assume that the effective field \vec{H}_{eff} is composed by external fields, i.e. a static field h_z and a circularly polarized driving field h_\perp as well as internal fields. The latter are an uniaxial anisotropy field $a S_z$ and an isotropic exchange field ($J > 0$).

$$\vec{H}_{eff} = (h_z + a S_z) \vec{e}_z + J \left(\frac{\partial^2 \vec{S}}{\partial x^2} \right) + h_\perp (\cos \omega t \vec{e}_x + \sin \omega t \vec{e}_y). \quad (2)$$

To get rid of the explicit time dependence we first transform to a rotating frame of reference, a fact which must not be forgotten when interpreting our results.

$$S'_x = S_x \cos(\omega t) + S_y \sin(\omega t), \quad S'_y = -S_x \sin(\omega t) + S_y \cos(\omega t), \quad S'_z = S_z. \quad (3)$$

Observing then that eq. (1) preserves the modulus of the local spin density $\vec{S}(x, t)$, we may put $|\vec{S}| = 1$. Eliminating the superfluous coordinate with the help of a stereographic projection

$$\Phi = \frac{S'_x + i S'_y}{1 + S'_z} \quad (4)$$

we obtain the final form for the equation of motion

$$\begin{aligned} \frac{\partial \Phi}{\partial t} = & (i - \Gamma) \left[(\delta + i \gamma) \Phi + a \Phi \left(\frac{1 - |\Phi|^2}{1 + |\Phi|^2} \right) - \frac{h_\perp}{2} (1 - \Phi^2) \right. \\ & \left. + J \left(\frac{2 \bar{\Phi}}{1 + |\Phi|^2} \left(\frac{\partial \Phi}{\partial x} \right)^2 - \frac{\partial^2 \Phi}{\partial x^2} \right) \right]. \end{aligned} \quad (5)$$

The parameters

$$\delta = h_z - \frac{\omega}{1 + \Gamma^2} \quad \text{and} \quad \gamma = \frac{\omega \Gamma}{1 + \Gamma^2} \quad (6)$$

denote the detuning from resonance and the renormalized external frequency, respectively.

Considering the set of parameters characterizing the effective field it seems reasonable to scale them in terms of the anisotropy $|a|$. Thus a takes on only the values ± 1 , distinguishing between the easy-axis (+1) and easy-plane (−1) case. Furthermore in eq. (5) the exchange constant J may be absorbed into the variable x .

Studying the bifurcation behaviour of this system we may confine ourselves to the case $h_\perp > 0$ and $\gamma > 0$. A negative value of h_\perp merely corresponds to shifting the phase of the driving field by π . Furthermore eq. (5) remains invariant under the transformation $(\Phi \rightarrow \frac{1}{\Phi}, \delta \rightarrow -\delta, \gamma \rightarrow -\gamma)$, implying that all bifurcation diagrams are symmetric with respect to the origin in the δ - γ plane.

Our analytical calculations of bifurcation lines, amplitude equations, etc. are performed without imposing any restrictions on the parameters Γ , h_\perp , δ and γ . For numerical purposes we will always choose $\Gamma = 0.1$ and keep h_\perp fixed at $h_\perp = 0.1$. This means that we investigate the bifurcation behaviour in the remaining δ - γ parameter plane. For the codimension-2 bifurcation, to be discussed later on, the variation of the coefficients of the amplitude equations will be given as a function of h_\perp also.

3 Spatially homogeneous fixed point solutions

We start our systematic investigation of eq. (5) by analyzing its spatially homogeneous fixed point solutions ($\partial_x \Phi_0 = \partial_t \Phi_0 = 0$). A rather complete bifurcation analysis – including codimension-3 bifurcations – of the spatially homogeneous system can be found in [7]. Thus we confine ourselves to a description of some elementary results which are important in the present context. The spatially homogeneous time independent solutions of eq. (5) for example are shown to be given by the roots of the algebraic equation:

$$(\gamma^2 + a^2) S_z^4 + 2a\delta S_z^3 + (\delta^2 + h_\perp^2 - a^2 - \gamma^2) S_z^2 - 2a\delta S_z - \delta^2 = 0. \quad (7)$$

Here S_z is the cartesian spin coordinate. From the solution of eq. (7) S_x and S_y (and therefore Φ) may be determined explicitly. Depending on the values of the parameters this polynomial has either two or four solutions. The corresponding regions in parameter space are separated by saddle-node bifurcation lines as is shown in figure 1. As the substitution $a \rightarrow -a$ and $S_z \rightarrow -S_z$ leaves eq. (7) invariant, this bifurcation does not depend on the type of anisotropy chosen. Although the saddle-node bifurcations can be calculated easily from eq. (7) we skip this calculation here because an equivalent expression is obtained within the framework of our stability analysis in the next section.

< Fig.1

4 Stability analysis

Let Φ_0 denote a spatially homogeneous fixed point solution of eq. (5). Proceeding as usual in discussing linear stability we add an infinitesimal perturbation $\delta\Phi(x, t)$

$$\Phi(x, t) = \Phi_0 + \delta\Phi(x, t). \quad (8)$$

Its time evolution is governed in first order by the equation

$$\delta\dot{\Phi} = (i - \Gamma) [g_1 - \partial_x^2 + i g_2 + (g_3 + i g_4) \mathcal{C}] \delta\Phi, \quad (9)$$

where \mathcal{C} denotes the operator of complex conjugation, i.e. $\mathcal{C}f = \bar{f}$ and g_1, \dots, g_4 are given by

$$g_1 = \delta + h_\perp \text{Re}(\Phi_0) + a \frac{1 - 2|\Phi_0|^2 - |\Phi_0|^4}{(1 + |\Phi_0|^2)^2} \quad (10)$$

$$g_2 = \gamma + h_\perp \text{Im}(\Phi_0) \quad (11)$$

$$g_3 = \frac{-2a}{(1 + |\Phi_0|^2)^2} \text{Re}(\Phi_0^2) \quad (12)$$

$$g_4 = \frac{-2a}{(1 + |\Phi_0|^2)^2} \text{Im}(\Phi_0^2). \quad (13)$$

The plane wave solutions of eq. (9) are written in the form

$$\delta\Phi_q = \cos(qx) \left(\delta\Phi_+ e^{\lambda t} + \delta\Phi_- e^{\bar{\lambda}t} \right), \quad (14)$$

which accommodates for the presence of the complex conjugation operator. This yields after some algebra the following secular equation

$$\lambda^2 - \lambda \operatorname{tr}(q) + \det(q) = 0, \quad (15)$$

with

$$\operatorname{tr}(q) = -2 (g_2 + \Gamma (g_1 + q^2)) \quad (16)$$

$$\det(q) = (1 + \Gamma^2) [(g_1 + q^2)^2 + g_2^2 - g_3^2 - g_4^2]. \quad (17)$$

Here tr and \det denote the trace and determinant of the two by two q -dependent coefficient matrix which arises when eq. (9) is split into real and imaginary part. As is obvious from their definitions $\{g_\alpha\}$, tr and \det depend on the fixed point coordinates $\Phi_0(\delta, \gamma, h_\perp, a = \pm 1)$ as well as on the parameters $(\delta, \gamma, h_\perp, a = \pm 1, \Gamma)$ and on the wavenumber q .

The solution Φ_0 is linearly stable if all possible perturbations $\delta\Phi_q$ decay exponentially in time, i.e. if $\operatorname{Re} \lambda_\pm(q) < 0$ for all q , where $\lambda_\pm(q)$ denotes the two roots of eq. (15) for some fixed wavenumber q . As is described in [8, 9] this yields the conditions

$$\operatorname{tr}(q) < 0 \quad \text{and} \quad \det(q) > 0 \quad (18)$$

which have to be fulfilled simultaneously for all q . Thus there are two different possibilities for these stability conditions to become violated, leading to soft-mode and hard-mode instabilities. When both of them are violated simultaneously we have a codimension-2 bifurcation which requires a separate discussion to be given later on.

4.1 Soft-mode instability

A soft-mode instability is characterized by a zero eigenvalue $\lambda_+(q_c) = 0$ which happens if $\det(q_c)$ vanishes. More precisely it occurs if the minimum value of $\det(q)$ becomes zero for some finite $q = q_c$, while $\operatorname{tr}(q)$ remains negative. Hence we get from eq. (17)

$$\partial_q \det|_{q=q_c} = (1 + \Gamma^2) 4 q_c (g_1 + q_c^2) = 0 \quad (19)$$

$$\det|_{q=q_c} = (1 + \Gamma^2) ((g_1 + q_c^2)^2 + g_2^2 - g_3^2 - g_4^2) = 0. \quad (20)$$

Thus for $q_c \neq 0$ we end up with the bifurcation condition

$$g_2^2 - g_3^2 - g_4^2 = 0 \quad (21)$$

defining a surface in the space of parameters. Its boundary results from $-g_1 = q_c^2 = 0$, which leads to the end points of the bifurcation lines in the δ - γ plane for Γ and h_\perp fixed.

The second solution $q_c = 0$ of eq. (19) just represents the spatially homogeneous saddle-node bifurcation mentioned in section 3. It is described by $g_1^2 + g_2^2 - g_3^2 - g_4^2 = 0$. The corresponding bifurcation lines have been depicted already in figure 1.

4.2 Hard-mode instability

In the case of a hard-mode instability the eigenvalues $\lambda(q_c)_\pm = \pm i\omega(q_c)$ become purely imaginary at the instability point. This occurs if $tr(q_c)$, more precisely its maximum value at some $q = q_c$, vanishes, while $\omega_c^2 = det(q_c)$ remains positive. From eq. (16) we get immediately

$$\partial_q tr|_{q=q_c} = -4 \Gamma q_c = 0, \quad i.e. \quad q_c = 0 \quad (22)$$

$$tr|_{q_c=0} = -2 (g_2 + \Gamma g_1) = 0. \quad (23)$$

Hence the bifurcation manifold is given by eq. (23) and its boundary by $\omega_c^2 = det = 0$.

The form of eq. (16) implies that the critical modes are spatially homogeneous. Thus there are no primary instabilities creating travelling wave states ($\omega_c \neq 0, q_c \neq 0$).

4.3 The bifurcation diagram

To obtain the soft- and hard-mode as well as saddle-node bifurcation lines in the space of the physical parameters we proceed as follows: Keeping $\Gamma = 0.1, a = \pm 1, h_\perp = 0.1$ fixed, we solve the two (real) fixed point equations –for convenience we use eq. (5), and not the polynomial (7)– together with the proper bifurcation condition numerically. Using the PITCON-package [10] we track the solution curves of this algebraic problem in the four dimensional space spanned by $Re(\Phi_0), Im(\Phi_0), \delta$ and γ . Its projections onto the δ - γ plane are the bifurcation lines sought for. For the special case of the spatially homogeneous saddle-node bifurcation the result has been presented already in figure 1. The soft- and hard-mode instabilities which complete this diagram are drawn in figures 2a and 2b for both types

< Fig.2

The gross features¹ of the solutions of our system following from figures 2a and 2b may be summarized as follows:

In the case of an easy-axis anisotropy (cf. fig. 2a) we find in region (I) of the parameter plane a stable solution near the north pole ($\Phi = 0$) and an unstable solution near the south pole ($\Phi = \infty$). Crossing the hard-mode bifurcation line from region (I) to region (II) the southern fixed point is stabilized and an unstable spatially homogeneous limit-cycle emerges. This limit-cycle in turn destabilizes the northern fixed point solution at the second hard-mode bifurcation line. Thus in region (III) we have a stable solution on the southern and an unstable solution on the northern hemisphere. Crossing the soft-mode bifurcation line separating regions (II) and (IV) the northern (stable) spatially homogeneous fixed point solution gets destabilized with regard to spatially periodic states.

For an easy-plane anisotropy (cf. fig. 2b) the situation in regions (I) and (III) is the same as in the easy-axis case. In region (II), however, both homogeneous fixed point solutions are

¹See reference [7] for more details concerning the spatially homogeneous system.

unstable. The northern fixed point of region (I) is now affected by the upper hard-mode line but not by the saddle-node bifurcation ($\overline{76}$). It loses stability with regard to spatially periodic perturbations when passing the soft-mode line separating regions (Ia) and (VI). Note that in accordance with the stability properties of the fixed point involved in the pitchfork bifurcation (6) two stable homogeneous states exist in region (Ia).

In this survey we have neither mentioned the fact, that on the lower HOPF line a transition from super- to subcritical bifurcations occurs implying saddle-node bifurcations of limit-cycles nor referred to the global bifurcations of the homogeneous system found in [7].

We stress, however, that our analysis leads via eq. (19) to an explicit expression for q_c along the soft-mode line. Starting from its origin at the homogeneous saddle-node bifurcation line q_c turns out to increase monotonously from zero to infinity. The corresponding results for ω_c are contained in [7] already.

We finally present in figure 3 a number of typical spectra for several fixed points at salient positions in the δ - γ plane.

< Fig.3

5 Amplitude equations

After having calculated the possible instabilities and the corresponding bifurcation lines in the space of parameters, we go on by analyzing the behaviour of our system in the neighbourhood of these marginal stability lines. This will be achieved with the help of a perturbation expansion. A natural expansion parameter ϵ measuring the distance from threshold results from writing

$$\delta = \delta_c + \epsilon^2 \delta_2 \quad (24)$$

$$\gamma = \gamma_c + \epsilon^2 \gamma_2. \quad (25)$$

(δ_c, γ_c) are the parameters at criticality whereas (δ_2, γ_2) , which may be thought of as the components of a unit vector, characterize the direction in which the bifurcation line is crossed transversally.

Let quite generally $u_0(x, t)$ denote the critical mode at instability. In order to apply perturbation theory above threshold we put as usual

$$\Phi = \Phi_0 + \epsilon A(\epsilon x, \epsilon^2 t) u_0(x, t) + \epsilon^2 r(x, t, \epsilon). \quad (26)$$

The scaling of the slow space and time coordinates

$$X = \epsilon x \quad \text{and} \quad T = \epsilon^2 t \quad (27)$$

is motivated by the quadratic dependence of the spectrum on the wavenumber q near criticality. The slowly varying amplitude $A(X, T)$ is determined so that the solution (26) contains no secular terms. Inserting the ansatz (26) into the equation of motion (5) this requirement leads via solvability conditions in lowest non-trivial order of perturbation theory to an equation governing the time evolution of A . This procedure is well known and has been often applied especially to hydrodynamic problems (cf. ref. [11, 12]). For a review

see the extensive paper by Cross and Hohenberg [2] as well as the book by Manneville [13]. Kuramoto and Tsuzuki [8, 14] used this technique in connection with reaction–diffusion equations and derived the coefficients of the amplitude equations for a large class of fundamental equations exhibiting soft- and hard-mode instabilities. Our equation of motion (5) does not fit into their scheme because of the quadratic derivative $(\frac{\partial \Phi}{\partial x})^2$. It will affect the nonlinear coefficient of the soft-mode amplitude equation.

Concerning magnetic systems this procedure was first applied by Elmer in the context of ferromagnetic resonance. His early work is based on a Landau-Lifshitz equation containing a dipolar interaction term whereas dissipative effects are taken into account employing the Bloch-Bloembergen damping mechanism [4]. More recently he described pattern formation in thin magnetic films studying a Landau-Lifshitz equation with Gilbert damping [15]. In contrast to our work he is interested mainly in the effects caused by the dipolar interaction.

The structure of the amplitude equations is determined completely by the nature of the instability considered and the symmetries of the underlying system. They are thus universal, usually turn out to be real or complex Ginzburg–Landau equations and are found in a wide variety of physical systems (see for example the numerous references in [2]). The individual physics of the system at hand manifests itself in the explicit values of their coefficients, more precisely in their dependence on the physical parameters. We aim at giving a fairly complete survey in this matter for our magnetic problem.

In the following section we present our results for the amplitude equations for the various instabilities and plot the variation of their coefficients along the corresponding bifurcation lines. As the necessary calculations in deriving these coefficients explicitly are very lengthy they are deferred to appendix A.

It has been pointed out (see ref. [2] and the references therein) that the linear coefficients of the amplitude equations (cf. eq.(31,32,33)) can be derived alternatively by differentiating the spectrum of the linearized equation of motion. In our case this amounts to –cf. eq. (9,15)–

$$\mu = \left. \frac{\partial \lambda}{\partial \gamma} \right|_c \gamma_2 + \left. \frac{\partial \lambda}{\partial \delta} \right|_c \delta_2 \quad (28)$$

$$\alpha = -\frac{1}{2} \left. \frac{\partial^2 \lambda}{\partial q^2} \right|_c. \quad (29)$$

The subscript means that these expressions have to be calculated at the critical values. Note that the linear coefficient μ depends on the direction in which the bifurcation line in the δ - γ plane is crossed when entering the unstable region. We will always choose it to be normal to that line. The diffusion coefficient α marks the dissipative part of the amplitude equation, whereas the nonlinear coefficient r characterizes the type of bifurcation, i.e. for example whether it is super- or subcritical, and fixes the saturation value of the amplitude in the supercritical case.

5.1 Soft-mode instability

For a soft-mode instability the slowly varying amplitude $A(X, T)$ has been introduced in appendix A according to

$$\Phi - \Phi_0 = \epsilon v_0 \left(A(\epsilon x, \epsilon^2 t) e^{i q_c x} + c.c. \right). \quad (30)$$

Its time evolution is found to be governed by a Ginzburg-Landau equation [8, 11, 12] with real coefficients:

$$\partial_T A = \mu_a A + \alpha_a \partial_X^2 A - r_a |A|^2 A. \quad (31)$$

This equation possesses a Lyapunov potential and is therefore of relaxational type. As the modulus of all of its coefficients may be rescaled to unity by appropriate redefinitions of time, space and amplitude, the most interesting traits of this equation show up in situations where these coefficients change sign and as a consequence the behaviour of the solutions changes qualitatively.

The variation of the coefficients along the bifurcation line is shown in figure 4 for the easy axis case. The diffusion coefficient α_a , which is proportional to q_c^2 , is zero at its origin which lies on the saddle-node bifurcation line and increases monotonously. At the origin itself the derivation of the amplitude equation loses its validity, an issue which will be discussed separately. Notice that the third-order coefficient changes sign along the bifurcation line signalling a transition from a subcritical to a supercritical bifurcation. In the supercritical case we have a (forward) bifurcation of a stable spatially periodic state, whereas for subcritical behaviour the system shows a (backward) bifurcation of an unstable spatially periodic state. Whether or not starting from a perturbation theoretical treatment in a neighbourhood of this degenerated point higher order terms lead to a stabilization remains undecided. In view of appendix A such calculations seem to be impracticable however, so that probably one will have to resort the full eq. (5) again to answer such questions. Work in this direction will be reported.

< Fig. 4

In order to achieve some qualitative understanding of this transition we have performed direct numerical simulations on the equation of motion (1) using parameter values in the neighbourhood of this point. Crossing the soft-mode bifurcation line at point A (cf. fig. 5) in the supercritical region ($r > 0$) a periodic state emerges continuously. Pursuing that solution along the curve C we find that it continues to exist with finite amplitude when point B on the subcritical side is reached. So, beyond point B a stable homogeneous and a stable periodic solution coexist. We conjecture that –in close analogy to degenerated HOPF bifurcations– there is a line of saddle node bifurcations of spatially periodic solutions on which the stable periodic solution is destroyed together with the unstable periodic one which is created when crossing the soft-mode bifurcation line at point B.

< Fig. 5

We have not plotted the coefficients as functions of the parameters for negative anisotropies along the segment of the soft-mode bifurcation line connecting the (homogeneous) saddle-node and the hard-mode bifurcation lines because this bifurcation is subcritical throughout. Recall that all these lines affect one and the same solution. Numerical simulations reveal that the solution merely relaxes towards the other stable (spatially homogenous) state after a transient period during which periodic structures appear.

The obvious divergences of the linear and nonlinear coefficients μ and r are rooted in the fact that the critical wavenumber q_c vanishes, when the soft-mode bifurcation approaches the saddle-node bifurcation. Firstly, the fixed point equations, depending on the parameters δ and γ , have to be expanded with respect to ϵ^2 in a neighbourhood of the critical parameter values. This can be done consistently only if the corresponding Jacobian has non-zero eigenvalues. At a saddle-node bifurcation this condition is violated and the expansion breaks down leading to a diverging behaviour for the coefficient μ . Secondly, as far as the nonlinear coefficient r is concerned, we have to solve an equation second order in ϵ . (Refer to appendix A for details). The corresponding solution loses its validity however for vanishing critical wavenumber because the linear operator on the left hand side simply reduces to the Jacobian so that trivially the solvability condition is no longer satisfied. Thus at this point in parameter space the formalism used to derive the amplitude equation becomes invalid. A similar situation has been observed by Kuramoto et. al. in [8].

5.2 Hard-mode instability

The amplitude equation for this type of instability is given by

$$\partial_T B = (\mu_{bR} + i \mu_{bI}) B + \alpha_b (1 + i c_1) \partial_X^2 B - r_b (1 - i c_2) |B|^2 B. \quad (32)$$

In contrast to the soft-mode case this equation has complex-valued coefficients. The real part α_b of the diffusion coefficient is identical to the damping constant Γ as is proved in appendix A, eq. (98). The imaginary part μ_{bI} of the linear coefficient can be removed by a transformation to a rotating frame ($B \rightarrow B e^{i \mu_{bI} t}$).

This complex Ginzburg-Landau equation has been addressed to frequently in the literature and is analysed in a wide variety of contexts. The behaviour of its solutions ranges from relaxational dynamics for vanishing imaginary parts of the coefficients ($c_1, c_2 = 0$) over some very complex – even chaotic – types of spatio-temporal patterns in an intermediate range of (c_1, c_2) to the completely integrable limit ($c_1, c_2 \rightarrow \infty$) of the nonlinear Schrödinger equation.

For easy-axis anisotropy the coefficients are plotted as functions of δ along the whole lower hard-mode bifurcation line (cf. figure 2a) in figure 6. The singularities appearing when the bifurcation line approaches the saddle-node bifurcation are caused by the vanishing of the critical frequency ω_c . Their explanation is completely analogous to that in the soft-mode case for vanishing q_c in the last section. Regarding the coefficient r_b we recognize a change of sign signalling a transition from a subcritical to a supercritical bifurcation. Unfortunately this degeneracy is found in a region of parameter space, where the corresponding solution Φ_0 is already unstable due to the soft-mode instability separating regions (II) and (IV). If the dependence on the driving field h_\perp is taken into account additionally, this situation changes as will be expounded in the following section in which the codimension-2 bifurcation is treated.

As has been demonstrated in [7] for the spatially homogeneous system explicitly, and is well known under general circumstances [16], such degeneracy of a hard-mode bifurcation entails a saddle-node bifurcation of (spatially homogeneous) limit-cycles. The resulting

< Fig. 6

bifurcation line ends at that point of the hard-mode instability where the coefficient r_b vanishes.

Along the upper hard-mode line (cf. fig. 2a) the bifurcation is subcritical everywhere. Hence the variation of the coefficients is not shown explicitly. The analysis of the spatially homogeneous system [7] makes evident that this subcritical bifurcation leads to the unstable, spatially homogeneous limit-cycle existing in region (II) of parameter space.

Concerning the easy-plane anisotropy, the variation of the coefficients for both hard-mode lines is depicted in figure 7. Along the upper curve (cf. fig. 7a) the bifurcation is supercritical for all values of the parameters whereas along the lower bifurcation line we recognize a transition from a sub- to a supercritical bifurcation (cf. fig. 7b). Regarding the supercritical portion of that line, the small numerical value of the coefficient r_b implies a large saturation amplitude of the oscillatory solution. We surmise that this is brought about by the stereographic projection technique, which leads to a large modulus of Φ_0 if the fixed point in question lies on the southern hemisphere. It is thus an artificial product.

< Fig. 7

5.3 Codimension-2 bifurcation

We now focus on the codimension-2 point in parameter space where the hard- and soft-mode bifurcations fall together. The spectrum of the linear operator at this point is depicted in figure 3 (1) and indicates that the degenerated critical mode is now composed of a spatially periodic time-independent and a time-periodic spatially homogeneous part. Supplying both of them with slowly varying amplitudes $A(X, T)$ and $B(X, T)$, respectively, we prove in appendix A.3 that their dynamics is governed by two coupled (real and complex) Ginzburg-Landau equations

$$\begin{aligned}\partial_T A &= \mu_a A + \alpha_a \partial_X^2 A - r_a |A|^2 A - s_a |B|^2 A \\ \partial_T B &= (\mu_{bR} + i \mu_{bI}) B + \alpha_b (1 + i c_1) \partial_X^2 B - r_b (1 - i c_2) |B|^2 B \\ &\quad - (s_{bR} + i s_{bI}) |A|^2 B.\end{aligned}\tag{33}$$

The linear coefficients as well as the nonlinear coefficients r_a , r_b and c_2 are the same as those in the uncoupled equations for the corresponding instabilities. This is obvious from continuation arguments. Only the coupling coefficients s_a and $s_b = s_{bR} + i s_{bI}$ turn out to be qualitatively new quantities. To get an impression of their behaviour we study their functional dependence of the amplitude h_\perp of the transversal driving field which was fixed up to now (cf. fig. 8). We desist from discussing the linear coefficients μ_a and μ_b because it is inevitable to pay attention to the actual direction in which the unstable region is entered in this case (cf. fig. 9). This was unnecessary hitherto, because the direction normal to the bifurcation line has been a natural choice.

< Fig. 8

For both values ± 1 of the anisotropy we recognize that the sign of the coefficient r_b changes for a certain amplitude h_\perp . Thus for this value, giving rise to a bifurcation of even higher codimension, the point of degeneracy of the hard-mode bifurcation, separating its sub- and supercritical forms, crosses the codimension-2 point where the hard- and soft-mode instabilities coincide.

For still higher amplitudes h_{\perp} – and easy-axis anisotropy ($a = +1$) – an interesting and complex situation arises which is illustrated schematically in figure 9. The degenerate hard-mode bifurcation D enters region (II-III, fig. 2a) so that the stable homogeneous (north pole) solution may lose stability not only in a subcritical HOPF bifurcation (left to D) but also in a supercritical one (between D and C).

< Fig. 9

In a neighbourhood of the codimension-2 point C there are two completely different limiting cases concerning the behaviour of the system. The first one has to be described (cf. region IV) by the real equation for the amplitude of the spatially periodic state, the second one (cf. region II) by the complex equation –with all its implications– governing the amplitude of the time-periodic mode. Considering μ_a and μ_b as functions of the angle Φ the coupled system of equations (33) provides a smooth transition between these dynamically radically different situations. There are furthermore saddle-node bifurcations of spatially homogeneous limit-cycles on a line emanating from D. We will tackle this problem among other topics in a subsequent paper.

Note that in the case of easy-plane anisotropy the codimension-2 bifurcation always involves a subcritical soft-mode bifurcation whereas it is supercritical for easy-axis anisotropy.

6 Summary

The first goal of the present paper has been to discuss systematically the instabilities of the spatially homogeneous states of a strongly driven ferromagnet against inhomogeneous perturbations. Based on the work by [7], in which only homogeneous magnetizations were treated, we have determined analytically the stability boundaries of all the fixed point solutions in the spatially extended system. Apart from the (spatially homogeneous) saddle-node and pitchfork bifurcations we have found hard-mode ($\omega_c \neq 0, q_c = 0$) and soft-mode ($\omega_c = 0, q_c \neq 0$) instabilities as well as codimension-2 situations where both instabilities coincide. The corresponding bifurcation lines have been presented in a δ - γ parameter plane at fixed h_{\perp} for easy-axis and easy-plane anisotropy.

We then went on to derive the amplitude equations near those instabilities which ultimately lead to pattern formation and have calculated all of their coefficients up to third order explicitly. Their functional dependence on the system parameters was shown along each bifurcation line which possesses at least one supercritical segment. From this variation the locations in parameter space where the real part of the third-order coefficients vanishes have been determined for both types of instability. At these points the bifurcation behaviour changes from sub- to supercritical. For a HOPF bifurcation such a degenerated situation is connected with a saddle-node bifurcation of limit-cycles. The respective bifurcation line in parameter space has been calculated already in [7]. As explained in section 5.1 we expect an analogous behaviour involving saddle-node bifurcations of spatially periodic states in the soft-mode case.

For the interesting and more involved case of the codimension-2 bifurcation we have found two coupled Ginzburg-Landau equations, one with real coefficients for the soft-mode amplitude and another one with complex coefficients for the hard-mode amplitude. This set

of equations describes a smooth transition between two fundamentally different limiting cases. The first one is given by the relaxational dynamics of the real Ginzburg-Landau equation for the amplitude of the spatially periodic state. The second one consists of the spatio-temporal dynamics described by the complex Ginzburg-Landau equation governing the amplitude of the time-periodic mode. In an intermediate range (cf. region (III) of figure 9) a competition between both behaviours will occur. A pretty complex situation, namely a bifurcation of even higher codimension, arises if the value of h_{\perp} is chosen in such a way that the degenerated HOPF bifurcation falls together with the codimension-2 point.

We emphasize that it was the explicit though laborious determination of the coefficients as functions of the physical parameters which has revealed the existence of these rather intriguing situations. As stated in the text there are other ones deserving closer attention. We just mention two of them. The first one is the saddle-node bifurcation of spatially periodic solutions, which is to be expected in the vicinity of the degenerated soft-mode bifurcation. The second one arises at the end points of our bifurcation lines. At these points, representing further examples of bifurcations of higher codimension, the perturbation theoretical ansatz including the scaling of space and time will have to be modified substantially. We are working on these problems, hoping that our results will be interesting not only from a more principal point of view, but –in revealing new aspects of pattern formation in magnetically ordered substances– also under realistic experimental conditions.

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Appendix A: Derivation of the envelope equations

This appendix contains the calculations of the explicit expressions for the coefficients of the amplitude equations starting from the basic equation of motion (5). Considering their extent they are confined to a reasonable measure. As a general reference for the procedure applied see for example [2, 8, 13]. For the purpose of this section it is convenient to formally rewrite the equation of motion using the substitution $\Phi = \Phi_0 + \delta\Phi$ and separate the linear and nonlinear contributions as

$$\hat{\Gamma} \delta\Phi = (i - \Gamma) N[\delta\Phi]. \quad (34)$$

$N[\delta\Phi]$ contains all the nonlinear terms in $\delta\Phi$ while $\hat{\Gamma}$ which is given by eq. (9) collects the linear ones. Both of them depend on the fixed point Φ_0 .

As has been exposed in section 5 we introduce a small parameter ϵ in order to study the system in the neighborhood of the marginal stability line by writing the parameters near

their critical values (δ_c, γ_c) as

$$\delta = \delta_c + \epsilon^2 \delta_2 \quad (35)$$

$$\gamma = \gamma_c + \epsilon^2 \gamma_2. \quad (36)$$

The solution of eq. (34) may then be assumed to be of the form

$$\delta \Phi = \epsilon \Phi_1 + \epsilon^2 \Phi_2 + \epsilon^3 \Phi_3. \quad (37)$$

The function Φ_1 contains the slowly varying amplitude $A(X, T)$ together with the basic mode becoming unstable. In the case of a soft-mode instability, for example, it is given by

$$\Phi_1 = v_0 (A(X, T) e^{i q_c x} + c.c.), \quad (38)$$

where v_0 is a complex number of modulus 1. Considering first the linear part of eq. (34) several expansions have to be performed: Because the fixed point coordinates Φ_0 depend on the parameters δ and γ via eq. (7) we have to expand them with respect to ϵ^2

$$\Phi_0 = \Phi_{00} + \epsilon^2 \Phi_{02}. \quad (39)$$

The second order contributions are determined by

$$\left. \frac{df}{d\epsilon^2} \right|_{\epsilon=0} = \frac{\partial f}{\partial \Phi_0} \Phi_{02} + \frac{\partial f}{\partial \bar{\Phi}_0} \bar{\Phi}_{02} + \frac{\partial f}{\partial \delta} \delta_2 + \frac{\partial f}{\partial \gamma} \gamma_2 = 0, \quad (40)$$

where f is an abbreviation for the bracket on the right hand side of eq. (5), disregarding the spatial derivatives. This leads to

$$\begin{aligned} & \Phi_{00} (\delta_2 + i \gamma_2) + \Phi_{02} (\delta_c + i \gamma_c + h_\perp \Phi_{00}) + \\ & a \frac{\Phi_{02} (1 - 2 |\Phi_{00}|^2 - |\Phi_{00}|^4) - 2 \bar{\Phi}_{02} \Phi_{00}^2}{(1 + |\Phi_{00}|^2)^2} = 0. \end{aligned} \quad (41)$$

The coefficients $\{g_1, \dots, g_4\}$ appearing in the operator $\hat{\Gamma}$ depend on the parameters as well as on the fixed point coordinates, which themselves are functions of these parameters. Thus the second order quantities in

$$g_\alpha = g_\alpha^{(0)} + \epsilon^2 g_\alpha^{(2)}, \quad \alpha = 1, \dots, 4, \quad (42)$$

are found to be:

$$g_1^{(2)} = \delta_2 + h_\perp \operatorname{Re}(\Phi_{02}) - \frac{4a}{(1 + |\Phi_{00}|^2)^3} (\bar{\Phi}_{00} \Phi_{02} + \Phi_{00} \bar{\Phi}_{02}) \quad (43)$$

$$g_2^{(2)} = \gamma_2 + h_\perp \operatorname{Im}(\Phi_{02}) \quad (44)$$

$$g_3^{(2)} = \frac{4a}{(1 + |\Phi_{00}|^2)^3} \operatorname{Re}(-\Phi_{00} \Phi_{02} + \Phi_{00}^3 \bar{\Phi}_{02}) \quad (45)$$

$$g_4^{(2)} = \frac{4a}{(1 + |\Phi_{00}|^2)^3} \operatorname{Im}(-\Phi_{00} \Phi_{02} + \Phi_{00}^3 \bar{\Phi}_{02}). \quad (46)$$

Finally the introduction of slow space $X = \epsilon x$ and time $T = \epsilon^2 t$ coordinates leads to the substitutions

$$\partial_x \rightarrow \partial_x + \epsilon \partial_X, \quad \partial_t \rightarrow \partial_t + \epsilon^2 \partial_T. \quad (47)$$

Putting all this together we obtain

$$\hat{\Gamma} = \hat{\Gamma}_0 + \epsilon \hat{\Gamma}_1 + \epsilon^2 \hat{\Gamma}_2, \quad (48)$$

with

$$\hat{\Gamma}_0 = \partial_t - (i - \Gamma) \left(g_1^{(0)} - \partial_x^2 + i g_2^{(0)} + (g_3^{(0)} + i g_4^{(0)}) \mathcal{C} \right) \quad (49)$$

$$\hat{\Gamma}_1 = 2(i - \Gamma) \partial_x \partial_X \quad (50)$$

$$\hat{\Gamma}_2 = \partial_T - (i - \Gamma) \left(g_1^{(2)} + i g_2^{(2)} + (g_3^{(2)} + i g_4^{(2)}) \mathcal{C} \right). \quad (51)$$

Similarly the nonlinear part of eq. (34) can be expanded as:

$$N[\delta\Phi] = \epsilon^2 N_2 + \epsilon^3 N_3 \quad (52)$$

with

$$N_2 = -\frac{2a}{(1 + |\Phi_{00}|^2)^3} [\bar{\Phi}_{00} \Phi_1^2 + 2\Phi_{00} |\Phi_1|^2 - \Phi_{00}^3 \bar{\Phi}_1^2] + \frac{h_\perp}{2} \Phi_1^2 + \frac{2\bar{\Phi}_{00}}{1 + |\Phi_{00}|^2} \left(\frac{\partial \Phi_1}{\partial x} \right)^2 \quad (53)$$

$$N_3 = -\frac{4a}{(1 + |\Phi_{00}|^2)^3} [\bar{\Phi}_{00} \Phi_1 \Phi_2 + \Phi_{00} (\bar{\Phi}_1 \Phi_2 + \Phi_1 \bar{\Phi}_2) - \Phi_{00}^3 \bar{\Phi}_1 \bar{\Phi}_2] + h_\perp \Phi_1 \Phi_2 + \quad (54)$$

$$\frac{2a}{(1 + |\Phi_{00}|^2)^4} [(2|\Phi_{00}|^2 - 1) \bar{\Phi}_1 \Phi_1^2 + \bar{\Phi}_{00}^2 \Phi_1^3 + 3\Phi_{00}^2 \Phi_1 \bar{\Phi}_1^2 - \Phi_{00}^4 \bar{\Phi}_1^3] +$$

$$\frac{4\bar{\Phi}_{00}}{1 + |\Phi_{00}|^2} \frac{\partial \Phi_1}{\partial x} \left(\frac{\partial \Phi_1}{\partial X} + \frac{\partial \Phi_2}{\partial x} \right) + 2 \frac{\bar{\Phi}_1 - \bar{\Phi}_{00}^2 \Phi_1}{(1 + |\Phi_{00}|^2)^2} \left(\frac{\partial \Phi_1}{\partial x} \right)^2.$$

Inserting the expansions (37,48,52) into eq. (34) yields in ascending orders of ϵ :

$$\hat{\Gamma}_0 \Phi_1 = 0 \quad (55)$$

$$\hat{\Gamma}_0 \Phi_2 = -\hat{\Gamma}_1 \Phi_1 + (i - \Gamma) N_2 \quad (56)$$

$$\hat{\Gamma}_0 \Phi_3 = -\hat{\Gamma}_2 \Phi_1 - \hat{\Gamma}_1 \Phi_2 + (i - \Gamma) N_3. \quad (57)$$

Observe that $\hat{\Gamma}_0$, containing a linear and an antilinear part, is a linear operator when acting on real-valued functions. In order to solve this hierarchy of equations we notice that the solution of the first equation is given by the critical mode itself (neglecting transient contributions). This implements a certain definite wavenumber and/or frequency into the theory and as a consequence the right hand side of these equations, depending on powers of this fundamental solution, can be expanded with respect to harmonics of this mode. In this way the solution of the problem may be constructed within a space of strictly periodic functions in time and/or space. It is established by applying conventional Fredholm theory which necessitates the introduction of an appropriate scalar product. An apt choice proves to be

$$(f, g) := \text{Re} \left(\frac{1}{\Lambda_c} \int_0^{\Lambda_c} dx \frac{1}{T_c} \int_0^{T_c} dt \bar{f}(x, t) g(x, t) \right) \quad (58)$$

with $T_c = \frac{2\pi}{\omega_c}$ and $\Lambda_c = \frac{2\pi}{q_c}$. For a soft-mode instability the critical mode is time independent and eq. (58) reduces to an integration over x , whereas for a hard-mode instability we have to deal only with a time-integration because there is no (fast) spatial coordinate.

In all three cases to be discussed the solvability condition explained below is satisfied automatically for eq. (56). The amplitude equations are obtained from the third-order eq. (57) which is solvable only if

$$0 = -(\Psi_0, \hat{\Gamma}_2 \Phi_1) - (\Psi_0, \hat{\Gamma}_1 \Phi_2) + (\Psi_0, (i - \Gamma) N_3). \quad (59)$$

Ψ_0 is the left-null-eigenvector of $\hat{\Gamma}_0$ (which is of course the same as the right-null-eigenvector of $\hat{\Gamma}_0^+$). This condition, also known as the Fredholm alternative, assures that the right hand side of eq. (57) contains no contribution exciting an eigenvector of $\hat{\Gamma}_0$ with zero eigenvalue. The ensuing construction guarantees, that no secular terms occur in the perturbation expansion. For further details see [2] and appendix A.2 of [13].

We now split up our discussion which up to now has been quite general according to the different instabilities.

A.1 Soft-mode instability

From eq. (55) the critical mode is given by

$$\Phi_1 = v_0 (A e^{i q_c x} + c.c.), \quad (60)$$

$$v_0 = \frac{g_3^{(0)} + i(g_2^{(0)} + g_4^{(0)})}{\sqrt{(g_3^{(0)})^2 + (g_2^{(0)} + g_4^{(0)})^2}}. \quad (61)$$

The left-0-eigenvector of $\hat{\Gamma}_0$ reads

$$\Psi_0 = w_0 (D e^{i q_c x} + c.c.) \quad (62)$$

$$w_0 = (i - \Gamma) \frac{g_2^{(0)} + g_4^{(0)} - i g_3^{(0)}}{\sqrt{(g_3^{(0)})^2 + (g_2^{(0)} + g_4^{(0)})^2}} \quad (63)$$

where the (arbitrarily chosen) normalization

$$(w_0, v_0) = 1 \quad (64)$$

proves to be convenient. The complex parameter D represents an undetermined phase factor and is used later on for the extraction of the amplitude equation.

Inserting Φ_1 into the second order equation, the latter can be solved because there are no resonant terms. We get

$$\Phi_2 = 2 c_{20} |A|^2 + c_{22} (A^2 e^{2 i q_c x} + c.c.) - v_1 \frac{1}{g_2^{(0)}} \partial_x \partial_X (A e^{i q_c x} + c.c.) \quad (65)$$

with

$$v_1 = \frac{1}{\sqrt{(g_3^{(0)})^2 + (g_2^{(0)} + g_4^{(0)})^2}} \left(g_2^{(0)} + g_4^{(0)} - i g_3^{(0)} \right) \quad (66)$$

$$c_{20} = \frac{1}{q_c^4} \left[(q_c^2 + i g_2^{(0)}) (n_{2a} + n_{2b}) + (g_3^{(0)} + i g_4^{(0)}) (\bar{n}_{2a} + \bar{n}_{2b}) \right] \quad (67)$$

$$c_{22} = \frac{1}{9 q_c^4} \left[(-3 q_c^2 + i g_2^{(0)}) (n_{2a} - n_{2b}) + (g_3^{(0)} + i g_4^{(0)}) (\bar{n}_{2a} - \bar{n}_{2b}) \right] \quad (68)$$

$$n_{2a} = \frac{h_{\perp}}{2} v_0^2 - \frac{2a}{(1 + |\Phi_{00}|^2)^3} [\bar{\Phi}_{00} v_0^2 + 2 \Phi_{00} |v_0|^2 - \Phi_{00}^3 \bar{v}_0^2] \quad (69)$$

$$n_{2b} = \frac{2 \bar{\Phi}_{00} v_0^2 q_c^2}{1 + |\Phi_{00}|^2}. \quad (70)$$

There is an additional, arbitrary contribution proportional to Φ_1 which belongs to the nullspace of the linear operator $\hat{\Gamma}_0$. It is not considered any further because it does not influence the amplitude equation.

Putting Φ_1 and Φ_2 into the third order eq. (57), we finally can exploit the solvability condition (59). Gathering all terms proportional to \bar{D} we end up with a Ginzburg-Landau equation with real coefficients:

$$\partial_T A = \mu_a A + \alpha_a \partial_X^2 A - r_a |A|^2 A. \quad (71)$$

Analogously the terms multiplying D yield the complex conjugate equation. The coefficients are given explicitly by

$$\mu_a = -\frac{1 + \Gamma^2}{g_2^{(0)}} \left[g_2^{(0)} g_2^{(2)} - g_3^{(0)} g_3^{(2)} - g_4^{(0)} g_4^{(2)} \right] \quad (72)$$

$$\alpha_a = 2 \frac{1 + \Gamma^2}{g_2^{(0)}} q_c^2 \quad (73)$$

$$r_a = \frac{1 + \Gamma^2}{\sqrt{(g_3^{(0)})^2 + (g_2^{(0)} + g_4^{(0)})^2}} \operatorname{Re} \left((g_2^{(0)} + g_4^{(0)} + i g_3^{(0)}) (n_{3a} + n_{3b}) \right). \quad (74)$$

with

$$n_{3a} = h_{\perp} v_0 (2 c_{20} + c_{22}) - \quad (75)$$

$$\begin{aligned} & \frac{4a}{(1 + |\Phi_{00}|^2)^3} [(\bar{\Phi}_{00} v_0 + \Phi_{00} \bar{v}_0) (2 c_{20} + c_{22}) + (\Phi_{00} v_0 - \Phi_{00}^3 \bar{v}_0) (2 \bar{c}_{20} + \bar{c}_{22})] + \\ & \frac{6a}{(1 + |\Phi_{00}|^2)^4} [(2 |\Phi_{00}|^2 - 1) \bar{v}_0 v_0^2 + \bar{\Phi}_{00}^2 v_0^3 + 3 \Phi_{00}^2 v_0 \bar{v}_0^2 - \Phi_{00}^4 \bar{v}_0^3] \\ n_{3b} &= \frac{8 \bar{\Phi}_{00} v_0 q_c^2}{1 + |\Phi_{00}|^2} c_{22} + \frac{2 q_c^2 v_0^2}{(1 + |\Phi_{00}|^2)^2} (\bar{v}_0 - \bar{\Phi}_{00}^2 v_0). \end{aligned} \quad (76)$$

A.2 Hard-mode instability

The derivation of the amplitude equations for the hard-mode instability proceeds along the same lines as that for the soft-mode case: the space-periodic critical mode has to be replaced merely by a time-periodic one. The main difference between the two instabilities is caused by the symmetries of the basic equation of motion. For a soft-mode instability

the spatial inversion symmetry leads to a Ginzburg-Landau equation with real coefficients. In the hard-mode case complex coefficients are admissible because eq. (5) is not invariant under time reversal.

The linear mode at marginal stability is now given by:

$$\Phi_1 = v_{0a} (B e^{i \omega_c t} + c.c.) + v_{0b} i (B e^{i \omega_c t} - c.c.) \quad (77)$$

with

$$v_{0a} = \frac{1}{N_v} \left(g_3^{(0)} - \Gamma g_4^{(0)} - (1 + \Gamma^2) g_1^{(0)} + i (g_4^{(0)} + \Gamma g_3^{(0)}) \right) \quad (78)$$

$$v_{0b} = \frac{1}{N_v} i \omega_c \quad (79)$$

$$N_v = \sqrt{2 (1 + \Gamma^2) g_1^{(0)} \left((1 + \Gamma^2) g_1^{(0)} + \Gamma g_4^{(0)} - g_3^{(0)} \right)}. \quad (80)$$

The left-0-eigenvector of $\hat{\Gamma}_0$ reads

$$\Psi_0 = w_{0a} (E e^{i \omega_c t} + c.c.) - w_{0b} i (E e^{i \omega_c t} - c.c.), \quad (81)$$

where

$$w_{0a} = \frac{1}{N_w} \left(g_4^{(0)} + \Gamma g_3^{(0)} + i (-g_3^{(0)} + \Gamma g_4^{(0)} + (1 + \Gamma^2) g_1^{(0)}) \right) \quad (82)$$

$$w_{0b} = -\frac{\omega_c}{N_w} \quad (83)$$

$$N_w = \frac{\omega_c}{2 (1 + \Gamma^2) g_1^{(0)}} N_v, \quad (84)$$

and the following normalization has been chosen:

$$(w_{0a}, v_{0a}) = (w_{0b}, v_{0b}) = 0 \quad (85)$$

$$(w_{0a}, v_{0b}) = (w_{0b}, v_{0a}) = 1. \quad (86)$$

As before the second order eq. (56) has no resonant contributions and can be solved:

$$\Phi_2 = c_{2n} |B|^2 + c_{2p2} (B e^{i \omega_c t} + c.c.) + c_{2m2} i (B e^{i \omega_c t} - c.c.) \quad (87)$$

with

$$c_{2n} = \frac{1 + \Gamma^2}{\omega_c^2} \left[-g_1^{(0)} (1 + i \Gamma) n_{2n} + (g_3^{(0)} + i g_4^{(0)}) \bar{n}_{2n} \right] \quad (88)$$

$$c_{2m2} = \frac{-1}{3 \omega_c^2} \left[2 \omega_c (i - \Gamma) n_{2p2} + (1 + \Gamma^2) \left(i (i - \Gamma) g_1^{(0)} n_{2m2} + (g_3^{(0)} + i g_4^{(0)}) \bar{n}_{2m2} \right) \right] \quad (89)$$

$$c_{2p2} = \frac{-1}{3 \omega_c^2} \left[(1 + \Gamma^2) \left(i (i - \Gamma) g_1^{(0)} n_{2p2} + (g_3^{(0)} + i g_4^{(0)}) \bar{n}_{2p2} \right) - 2 \omega_c (i - \Gamma) n_{2m2} \right] \quad (90)$$

$$n_{2n} = h_{\perp} v_{0a}^2 + h_{\perp} v_{0b}^2 - \frac{4a}{(1 + |\Phi_{00}|^2)^3} (\bar{\Phi}_{00} v_{0a}^2 + 2\Phi_{00} v_{0a} \bar{v}_{0a} - \Phi_{00}^3 \bar{v}_{0a}^2 + \bar{\Phi}_{00} v_{0b}^2 + 2\Phi_{00} v_{0b} \bar{v}_{0b} - \Phi_{00}^3 \bar{v}_{0b}^2) \quad (91)$$

$$n_{2m2} = h_{\perp} v_{0a} v_{0b} - \frac{4a}{(1 + |\Phi_{00}|^2)^3} (\bar{\Phi}_{00} v_{0a} v_{0b} + \Phi_{00} \bar{v}_{0a} v_{0b} + \Phi_{00} v_{0a} \bar{v}_{0b} - \Phi_{00}^3 \bar{v}_{0a} \bar{v}_{0b}) \quad (92)$$

$$n_{2p2} = h_{\perp} \frac{v_{0a}^2 - v_{0b}^2}{2} - \frac{2a}{(1 + |\Phi_{00}|^2)^3} (\bar{\Phi}_{00} v_{0a}^2 + 2\Phi_{00} v_{0a} \bar{v}_{0a} - \Phi_{00}^3 \bar{v}_{0a}^2 - \bar{\Phi}_{00} v_{0b}^2 - 2\Phi_{00} v_{0b} \bar{v}_{0b} + \Phi_{00}^3 \bar{v}_{0b}^2) \quad (93)$$

In agreement with the soft-mode case we have dropped a term belonging to the kernel of $\hat{\Gamma}_0$. Using this solution the nonlinear contribution N_3 of eq. (57) can be expressed as

$$N_3 = |B|^2 (n_{3p1} (B e^{i\omega_c t} + c.c.) + n_{3m1} i (B e^{i\omega_c t} - c.c.)) + \sum_{\nu \neq \pm 1} n_{3\nu} e^{i\nu\omega_c t}. \quad (94)$$

We refrain from presenting the complete expressions for the coefficients n_{3p1} and n_{3m1} because they are rather lengthy. They are received by expanding the nonlinear part N_3 in terms of the real valued functions $(B e^{i\omega_c t} + c.c.)$ and $i(B e^{i\omega_c t} - c.c.)$. Using them the solvability condition for the third order eq. (59) can be evaluated. Again we obtain the amplitude equation by gathering all terms in front of \bar{E} :

$$\partial_T B = (\mu_{bR} + i\mu_{bI}) B + \alpha_b (1 + i c_1) \partial_X^2 B - r_b (1 - i c_2) |B|^2 B. \quad (95)$$

The coefficients of this complex Ginzburg-Landau equation are:

$$\mu_{bR} = -g_2^{(2)} - \Gamma g_1^{(2)} \quad (96)$$

$$\mu_{bI} = \frac{1 + \Gamma^2}{\omega_c} (g_1^{(0)} (g_1^{(2)} - \Gamma g_2^{(2)}) - g_3^{(0)} g_3^{(2)} - g_4^{(0)} g_4^{(2)}) \quad (97)$$

$$\alpha_b = \Gamma \quad (98)$$

$$\alpha_b \cdot c_1 = -\frac{(1 + \Gamma^2) g_1^{(0)}}{\omega_c} \quad (99)$$

$$r_b = -\frac{1}{2} [(w_{0a}, (i - \Gamma) n_{3m1}) + (w_{0b}, (i - \Gamma) n_{3p1})] \quad (100)$$

$$r_b \cdot c_2 = -\frac{1}{2} [(w_{0a}, (i - \Gamma) n_{3p1}) - (w_{0b}, (i - \Gamma) n_{3m1})]. \quad (101)$$

A.3 Codimension-2 point

In this case the bifurcation conditions (21) and (23) for a stationary ($\omega_c = 0, q_c \neq 0$) and an oscillatory ($\omega_c \neq 0, q_c = 0$) instability are satisfied simultaneously. This requirement distinguishes a single point (δ_c, γ_c) in the parameter plane and implies the relations

$$\omega_c^2 = (1 + \Gamma^2) q_c^4 \quad (102)$$

$$g_1^{(0)} = -q_c^2 \quad (103)$$

$$g_2^{(0)} = \Gamma q_c^2. \quad (104)$$

The critical mode is now composed of two degenerated modes and consequently one has to introduce two amplitudes $A(X, T)$ and $B(X, T)$ to describe their slow modulations, respectively:

$$\Phi_1 = v_0 (A(X, T)e^{i q_c x} + c.c.) + v_{0a} (B(X, T)e^{i \omega_c t} + c.c.) + v_{0b} i(B(X, T)e^{i \omega_c t} - c.c.). \quad (105)$$

v_0 , v_{0a} and v_{0b} are defined as previously, compare eq.s (61,78,79). The fact that two independent degenerated critical modes occur entails that two different left-0-eigenvectors enter the formulation of the solvability condition

$$\Psi_{0a} = w_0 (Ee^{i q_c x} + c.c.) \quad (106)$$

$$\Psi_{0b} = w_{0a} (De^{i \omega_c t} + c.c.) + w_{0b} i(De^{i \omega_c t} - c.c.). \quad (107)$$

E and D are arbitrary complex parameters. Because most of the following expressions are very lengthy, we do not present them in detail but restrict ourselves to just describing how to get them. Using a software package for symbolic mathematics it is straightforward to obtain them explicitly.

The notation being necessary to keep book of all the different time and space periodic functions and their coefficients is as follows: Introducing a real-valued basic set of functions, which consists of fourier modes and their harmonics

$$A_{n+}(x) = A^n e^{i n q_c x} + c.c. \quad (108)$$

$$A_{n-}(x) = i (A^n e^{i n q_c x} - c.c.) \quad (109)$$

$$B_{n+}(t) = B^n e^{i n \omega_c t} + c.c. \quad (110)$$

$$B_{n-}(t) = i (B^n e^{i n \omega_c t} - c.c.), \quad n = 1, 2, 3, \dots \quad (111)$$

all functions may be expanded with respect to them. Of course $A = A(X, T)$ and $B = B(X, T)$ still depend on the slow variables. Therefore by inserting in a first step the linear mode Φ_1 into the second-order eq. (56) its right hand side is built up by the host of functions

$$\{|A|^2, |B|^2, A_{1-}, A_{2+}, B_{2+}, B_{2-}, A_{1+} \cdot B_{1+}, A_{1+} \cdot B_{1-}\}. \quad (112)$$

Up to an unimportant term belonging to the nullspace of $\hat{\Gamma}_0$ the solution of this equation proves to be

$$\begin{aligned} \Phi_2 = & c_{2A}(0,0) |A|^2 + c_{2B}(0,0) |B|^2 + c_{2A}(2,0) A_{2+} + c_{2B+}(0,2) B_{2+} + \\ & c_{2B-}(0,2) B_{2-} + c_{2AB+}(1,1) A_{1+} B_{1+} + c_{2AB-}(1,1) A_{1+} B_{1-} + \\ & c_2(1,0) \partial_x \partial_X A_{1+}. \end{aligned} \quad (113)$$

Here the indices (μ, ν) of the coefficients $c_{2\dots}$ mark spatial and temporal fourier modes, respectively. They are calculated easily by inverting the linear operator on the sub-spaces spanned by our basic set of functions. Most of these calculations have been performed already in the sections dealing with the soft- and hard-mode instabilities. Only terms made up of products of spatial and temporal fourier modes require additional computations.

Putting Φ_1 and Φ_2 into the remaining third-order eq. (57) N_3 becomes

$$N_3 = n_{3A+}(1,0) |A|^2 A_{1+} + n_{3B+}(1,0) |B|^2 A_{1+} + n_{3B+}(0,1) |B|^2 B_{1+} + \quad (114)$$

$$n_{3B-}(0,1) |B|^2 B_{1-} + n_{3A+}(0,1) |A|^2 B_{1+} + n_{3A-}(0,1) |A|^2 B_{1-} + \dots \quad (115)$$

where the dots stand for all those modes which are nonresonant because they are either higher order harmonics or constants. Evaluating the solvability condition pertaining to eq. (57) for each of the two left-0-eigenvectors separately we obtain by collecting all terms belonging to \bar{E} or \bar{D} , respectively, the following equations governing the time-evolution of the amplitudes:

$$\begin{aligned}\partial_T A &= \mu_a A + \alpha_a \partial_X^2 A - r_a |A|^2 A - s_a |B|^2 A \\ \partial_T B &= (\mu_{bR} + i \mu_{bI}) B + \alpha_b (1 + i c_1) \partial_X^2 B - r_b (1 - i c_2) |B|^2 B - (s_{bR} + i s_{bI}) |A|^2 B.\end{aligned}\tag{116}$$

Hence we have found a Ginzburg-Landau equation with real coefficients for the amplitude A being coupled to a complex one for the amplitude B . The coupling coefficients are

$$s_a = -(w_0, (i - \Gamma) n_{3B+}(1, 0))\tag{117}$$

$$s_{bR} = -\frac{1}{2} [(w_{0a}, (i - \Gamma) n_{3A-}(0, 1)) + (w_{0b}, (i - \Gamma) n_{3A+}(0, 1))]\tag{118}$$

$$s_{bI} = \frac{1}{2} [(w_{0a}, (i - \Gamma) n_{3A+}(0, 1)) - (w_{0b}, (i - \Gamma) n_{3A-}(0, 1))].\tag{119}$$

All the other first and third-order coefficients are the same as in the soft- and hard-mode cases as is clear from continuation arguments.

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Figure captions

- Fig.1 Saddle-node bifurcation line of eq. (5) in the δ - γ parameter plane at $\Gamma = 0.1$ and $h_{\perp} = 0.1$. The number of spatially homogeneous fixed point solutions is indicated.
- Fig.2 Bifurcation diagrams including the hard- and soft-mode instabilities of eq. (5) at $\Gamma = 0.1$ and $h_{\perp} = 0.1$, (a) easy-axis, $a = +1$, (b) easy-plane, $a = -1$. The region near the homogeneous cusp point is magnified in the insert.
- Fig.3 Spectra of the linearized operator at various points of the bifurcation diagrams (cf.fig. 2). (1) degenerated soft- and hard-mode point, (2) homogeneous cusp point, (3) end point of the soft-mode line, (4) homogeneous cusp point, (5) Arnold-Takens-Bogdanov point, (6) homogeneous cusp point.
- Fig.4 Coefficients of the soft-mode amplitude equation for $a = +1$ on the bifurcation line $\overline{13}$ of fig. 2a. The coefficient r_a changes sign at $\delta \simeq -0.80$, $\gamma \simeq 0.18$.
- Fig.5 Schematic bifurcation diagram for the degenerated soft-mode instability. The labeling is explained in the text.
- Fig.6 Coefficients of the hard-mode amplitude equation for $a = +1$ on the lower bifurcation line of fig. 2a. The coefficients c_1 and c_2 are shown separately in the insert. r_b changes sign at $\delta \simeq -1.10$, $\gamma \simeq 0.036$.
- Fig.7 Coefficients of the hard-mode amplitude equation for $a = -1$, (a) on the upper hard-mode line, (b) on the lower hard-mode line of fig. 2b. The coefficients c_1 and c_2 are shown separately in the insert. r_b changes sign at $\delta \simeq -1.10$, $\gamma \simeq 0.036$. The region near zero is displayed on a smaller scale (of order 10^{-4}).
- Fig.8 Coefficients of the codimension-2 amplitude equations (a) for $a = +1$, (b) for $a = -1$. The coefficients c_1 , c_2 and α_a are shown separately in the insert. r_b changes sign at (a) $h_{\perp} \simeq 0.90$, $\delta \simeq -2.05$, $\gamma \simeq 0.21$, (b) $h_{\perp} \simeq 1.72$, $\delta \simeq -1.12$, $\gamma \simeq 0.31$.
- Fig.9 Schematic bifurcation diagram of the codimension-2 bifurcation and the degenerated hard-mode bifurcation at $a = +1$. The labeling is explained in the text.